On the Characterization of Pseudodifferential Operators (old and new)

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Abstract In the framework of the Weyl-Hörmander calculus, under a condition of "geodesic temperance", pseudodifferential operators can be characterized by the boundedness of their iterated commutators. As a corollary, functions of pseudodifferential operators are themselves pseudodifferential. Sufficient conditions are given for the geodesic temperance. In particular, it is valid in the Beals-Fefferman calculus.

Introduction

The first historical example of a characterization of pseudodifferential operators is due to R. Beals [1] for the following class of symbols

$$S^0_{0,0} = \left\{ a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^{n*}) \,\middle|\, \left|\partial_\xi^\alpha \partial_x^\beta a(x,\xi)\right| \leq C_{\alpha,\beta} \right\} .$$

An operator A can be written $A = a(x,D)$ with $a \in S^0_{0,0}$ if and only if A and its iterated commutators with the multiplications by x_j and the derivations $\partial/\partial x_j$ are bounded on L^2.

The aim of this paper is to give an analogous characterization for the more general classes of pseudodifferential operators occurring in the Beals-Fefferman calculus and the Weyl-Hörmander calculus. Such a characterization has important consequences:

- The Wiener property: if a pseudodifferential operator (of order 0) is invertible as an operator in L^2, its inverse is also a pseudodifferential operator.
- The compatibility with the functional calculus: C^∞ functions of pseudodifferential operator are themselves pseudodifferential.

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- It is a good starting point for a general theory of Fourier integral operators. We refer to [4] for that point which will not be developped here.

This paper gives a new presentation, with some simplifications and complements, of the results of [3] where it is shown that the characterization of pseudodifferential operator is valid under an assumption of *geodesic temperance*.

The new point of this paper is the section 3 which gives a sufficient condition, easy to check, for the *geodesic temperance*. In the framework of the Beals-Fefferman calculus, this condition is always satisfied, and so are its consequences.

1 Weyl-Hörmander calculus

We will denote by small latine letters (such as x) points of the configuration space $\mathbb{R}^n$, by greek letters (such as ξ) points of its dual space $(\mathbb{R}^n)^*$ and by capital letters ($X = (x,\xi)$) points of the phase space $\mathscr{X} = \mathbb{R}^n \times (\mathbb{R}^n)^*$. The space $\mathscr{X}$ is equipped with the symplectic form σ defined by

$$\sigma(X,Y) = \langle \eta, x\rangle - \langle \xi, y\rangle \quad \text{for} \quad X = (x,\xi) \text{ and } Y = (y,\eta)\ .$$

The Weyl quantization associates to a function or distribution a on $\mathscr{X}$ an operator $a^w(x,D)$ acting in $\mathbb{R}^n$ defined by

$$a^w(x,D)u(x) = \iint \mathrm{e}^{\mathrm{i}\langle x-y,\xi\rangle} a\left(\tfrac{x+y}{2},\xi\right) u(y)\,\mathrm{d}y\,\mathrm{d}\xi/(2\pi)^n\ .$$

Considered in a weak sense, for $a \in \mathscr{S}'(\mathscr{X})$, this formula defines $a^w(x,D)$ as an operator mapping the Schwartz space $\mathscr{S}(\mathbb{R}^n)$ into the space $\mathscr{S}'(\mathbb{R}^n)$ of tempered distributions. Conversely, for any such operator A, there is a unique $a \in \mathscr{S}'(\mathscr{X})$, the *symbol* of A, such that $A = a^w(x,D)$.

Definition 1. The Hörmander classes of symbol $S(M,g)$ are associated to

- a Riemannian metric g on $\mathscr{X}$, identified with an application $\big(X \mapsto g_X(\cdot)\big)$, where each g_X is a positive definite quadratic form on $\mathscr{X}$,
- a weight M, i.e. a positive function on $\mathscr{X}$.

They are defined by

$$S(M,g) = \left\{ a \in C^\infty(\mathscr{X}) \,\middle|\, \left|\partial_{T_1}\dots\partial_{T_k} b(X)\right| \le C_k M(X) \text{ for } k \ge 0 \text{ and } g_X(T_j) \le 1 \right\}.$$

We use the notation $\partial_T f(X) = \langle \mathrm{d}f(X), T\rangle$ for the directional derivatives.

Example 1 (Beals-Fefferman classes [2]*).*
For $Q, q \in \mathbb{R}$ and Φ, φ positive functions on $\mathscr{X}$, these classes are defined by

$$a \in S^{Q,q}_{\Phi,\varphi} \iff \left|\partial_\xi^\alpha \partial_x^\beta a(x,\xi)\right| \le C_{\alpha,\beta}\,\Phi(x,\xi)^{Q-|\alpha|}\varphi(x,\xi)^{q-|\beta|}\ .$$

They are actually classes $S(M,g)$ with

$$g_X(\mathrm{d}x,\mathrm{d}\xi) = \frac{\mathrm{d}x^2}{\varphi(X)^2} + \frac{\mathrm{d}\xi^2}{\Phi(X)^2} \quad \text{and} \quad M = \varphi^q \Phi^Q \,. \tag{1}$$

In the particular case:

$$\Phi(x,\xi) = (1+|\xi|^2)^{\rho/2}\,,\ \varphi(x,\xi) = (1+|\xi|^2)^{-\delta/2}\,,$$

one recovers the Hörmander class $S^m_{\rho,\delta}$ with $m = Q\rho - q\delta$:

$$a \in S^m_{\rho,\delta} \iff \left|\partial^\alpha_\xi \partial^\beta_x a(x,\xi)\right| \le C_{\alpha,\beta}(1+|\xi|)^{m-\rho|\alpha|+\delta|\beta|}\,.$$

1.1 Admissible metrics

Before stating the conditions which guarantee that the classes $S(M,g)$ give rise to a good symbolic calculus, we should recall some well known properties of quadratic forms in symplectic spaces.

Reduced form. For each $Y \in \mathscr{X}$, one can choose symplectic coordinates (x',ξ') (depending on Y) such that the quadratic form g_Y takes the diagonal form:

$$g_Y(\mathrm{d}x',\mathrm{d}\xi') = \sum_j \frac{\mathrm{d}{x'_j}^2 + \mathrm{d}{\xi'_j}^2}{\lambda_j(Y)}\,. \tag{2}$$

The λ_j are positive and are independent of the particular choice of (x',ξ'). An important invariant, which will characterize the "gain" in the symbolic calculus, is the following:

$$\lambda(Y) = \min_j \lambda_j(Y)\,.$$

Inverse metric. It is the Riemannian metric g^σ on $\mathscr{X}$, defined for each Y by $g^\sigma_Y(T) = \sup \frac{\sigma(T,S)^2}{g_Y(S)}$. If $\mathscr{X}$ is identified with its dual via the symplectic form, g^σ_Y is nothing but the inverse quadratic form g_Y^{-1} a priori defined on $\mathscr{X}'$. In the symplectic coordinates above, one has

$$g^\sigma_Y(\mathrm{d}x',\mathrm{d}\xi') = \sum \lambda_j(Y)(\mathrm{d}{x'_j}^2 + \mathrm{d}{\xi'_j}^2)\,.$$

Geometric mean of g and g^σ. It is a third Riemannian metric $g^\#$ on $\mathscr{X}$. For each Y the quadratic form $g^\#_Y$ is the geometric mean of g_Y and g^σ_Y (the geometric mean of two positive definite quadratic forms is always canonically defined and is easily computed in a basis which diagonalizes the two quadratic forms). In the symplectic coordinates above, one has

$$g_Y^\#(\mathrm{d}x',\mathrm{d}\xi') = \sum(\mathrm{d}x_j'^2 + \mathrm{d}\xi_j'^2) \ .$$

Definition 2. The metric g is said *without symplectic eccentricity* (WSE) if for any Y, all the $\lambda_j(Y)$ are equal to $\lambda(Y)$. It is equivalent to say that $g^\sigma = \lambda^2 g$.

Example 2. Metrics of Beals-Fefferman (1) are WSE, with $\lambda = \Phi\varphi$.

Definition 3. The metric g is said *admissible* if it satisfies the following properties:

(i) *Uncertainty principle:* $\lambda(Y) \geq 1$.
(ii) *Slowness:* $\exists C, \quad g_Y(X-Y) \leq C^{-1} \Longrightarrow \frac{g_Y(T)}{g_X(T)} \leq C$.
(iii) *Temperance:* $\exists C,N, \quad \frac{g_Y(T)}{g_X(T)} \leq C\,(1+g_Y^\sigma(X-Y))^N$.

A weight M is said admissible for g (or a g-weight) if it satisfies

(ii′) $\exists C, \quad g_Y(X-Y) \leq C^{-1} \Longrightarrow M(Y)/M(X) \leq C$.
(iii′) $\exists C,N, \quad M(Y)/M(X) \leq C\,(1+g_Y^\sigma(X-Y))^N$.

As a consequence of (iii), one has

$$(1+g_X^\sigma(X-Y)) \leq C\,(1+g_Y^\sigma(X-Y))^{N+1} \ , \tag{3}$$

and thus $g_X(T)/g_Y(T)$ is also bounded by the right hand side of (iii) (with different values of C and N).

Remark 1. For an admissible Beals-Fefferman metric (1), condition (i) means that $\Phi(X)\varphi(X) \geq 1$ while (ii) and (iii) express bounds of the ratios $(\Phi(Y)/\Phi(X))^{\pm 1}$ and $(\varphi(Y)/\varphi(X))^{\pm 1}$.

The classes $S^m_{\rho,\delta}$ correspond to an admissible metric if $\delta \leq \rho \leq 1$ and $\delta < 1$.

1.2 Symbolic calculus

In this paragraph, g will denote an admissible metric and M a g-weight. Let us denote by $\mathrm{Op}\,S(M,g)$ the class of operators whose symbol belongs to $S(M,g)$ and by # the composition of symbols which corresponds to the composition of operators, i.e.

$$(a_1\#a_2)^w(x,D) = a_1^w(x,D)\circ a_2^w(x,D) \ .$$

The following properties are classical [6].

- Operators in $\mathrm{Op}\,S(M,g)$ map $\mathscr{S}(\mathbb{R}^n)$ into itself, $\mathscr{S}'(\mathbb{R}^n)$ into itself and, for $M=1$, $L^2(\mathbb{R}^n)$ into itself.
- The formal adjoint of $a^w(x,D)$ is $\overline{a}^w(x,D)$.
- If M_1 and M_2 are two g-weights, then M_1M_2 is also a g-weight and one has $S(M_1,g)\#S(M_2,g) \subset S(M_1M_2,g)$.

- For $a_j \in S(M_j, g)$, one has the asymptotic expansion

$$a_1 \# a_2(X) = \sum_{k=0}^{N-1} \frac{1}{k!} \left\{ \left(\tfrac{1}{2\mathrm{i}} \sigma(\partial_Y, \partial_Z)\right)^k a_1(Y) a_2(Z) \right\}_{\big|_{Y=Z=X}} + R_N(X) \,,$$

  where the k^{th} term in the sum belongs to $S(M_1 M_2 \lambda^{-k}, g)$ and R_N belongs to $S(M_1 M_2 \lambda^{-N}, g)$. The first two terms are the usual product and the Poisson bracket:

$$a_1 \# a_2 = a_1 a_2 + \tfrac{1}{2\mathrm{i}} \{a_1, a_2\} + \cdots$$

1.3 Can one use only metrics without symplectic eccentricity?

As remarked above, Beals-Fefferman metrics are WSE and one can say that, since the introduction of the Weyl-Hörmander calculus thirty years ago, almost all the metrics which have been used are WSE. It is not uniquely because they are simpler to use, a result of J. Toft [7] shows that there is a good mathematical reason for this limitation.

Theorem 1 (J. Toft). (i) *If g is an admissible metric, one has*

$$\exists C, N, \quad (g_X(T)/g_Y(T))^{\pm 1} \leq C \left(1 + g_Y^{\#}(X - Y)\right)^N \,. \tag{4}$$

(ii) *The metric $g^{\#}$ is admissible.*

The second point is an easy consequence of the first one. The right hand side of (4), which controls the ratio g_X/g_Y, controls also the ratio of the inverse metric g_X^σ/g_Y^σ and thus the ratio of their geometric mean $g_X^{\#}/g_Y^{\#}$, which proves that $g^{\#}$ is tempered. It controls also the ratio $\lambda(Y)/\lambda(X)$.

Let us introduce a new metric: $\widetilde{g} = \lambda^{-1} g^{\#}$. The ratio $\widetilde{g}_Y/\widetilde{g}_X$ is then controlled by the right hand side of (4) (with different constants C and N) and a fortiori by a power of $\left(1 + \lambda(Y) g_Y^{\#}(X - Y)\right)$. This proves the temperance of $\widetilde{g}$, because $\widetilde{g}^\sigma = \lambda g^{\#}$. The slowness being evident, the metric $\widetilde{g}$ is admissible. Moreover, it is WSE: in the coordinates (2), one has

$$g_Y(\mathrm{d}x', \mathrm{d}\xi') = \sum_j \frac{\mathrm{d}x_j'^{\,2} + \mathrm{d}\xi_j'^{2}}{\lambda_j(Y)} \quad ; \quad \widetilde{g}_Y(\mathrm{d}x', \mathrm{d}\xi') = \sum_j \frac{\mathrm{d}x_j'^{\,2} + \mathrm{d}\xi_j'^{2}}{\lambda(Y)} \,.$$

Let us now compare the g-calculus and the $\widetilde{g}$-calculus.

(i). If M is a g-weight, then M is a $\widetilde{g}$-weight.
(ii). One has then $S(M, g) \subset S(M, \widetilde{g})$.
(iii). Moreover, the Sobolev spaces are the same: $H(M, g) = H(M, \widetilde{g})$.
(iv). Any $A \in \operatorname{Op} S(M, \widetilde{g})$ maps $H(M_1)$ into $H(M_1/M)$.
(v). The "gain" λ of the symbolic calculus is the same for g and $\widetilde{g}$.

The first point is a consequence of Toft's theorem, the ratio $M(Y)/M(X)$ being also controlled by the right hand side of (4). The second point is evident. The Sobolev space $H(M,g)$ can be defined as the space of $u \in \mathscr{S}'(\mathbb{R}^n)$ such that $a^w(x,D)u \in L^2$ for any $a \in S(M,g)$, which makes (iv) evident. Equivalent definitions can be found in [5], where the theorem 6.9 proves (iii).

It is thus difficult to imagine a situation where it would be more advantageous to use the g-calculus instead of the $\widetilde{g}$-calculus.

2 Characterization of pseudodifferential operators

2.1 Geodesic temperance

Let us denote by $d^\sigma(X,Y)$ the geodesic distance, for the Riemannian metric g^σ, between X and Y.

Definition 4. The metric g is said *geodesically tempered* if it is tempered and if, moreover, the equivalent following conditions are satisfied:

$$\exists C,N;\ \frac{g_Y(T)}{g_X(T)} \leq C\left(1+d^\sigma(X,Y)\right)^N , \tag{5}$$

$$\exists C,N;\ C^{-1}\left(1+d^\sigma(X,Y)\right)^{1/N} \leq \left(1+g_Y^\sigma(X-Y)\right) \leq C\left(1+d^\sigma(X,Y)\right)^N . \tag{6}$$

The left part of (6) is always true. Actually, one has

$$C^{-1}\left(1+g_Y^\sigma(X-Y)\right)^{1/N} \leq 1+g^\sigma\text{-length of segment } XY \leq C\left(1+g_Y^\sigma(X-Y)\right)^N ,$$

which is a simple consequence of (3) and thus of the temperance of g. It is clear that the right part of (6) imply (5).

Assume now (5), and let $t \in [0,L] \mapsto X(t)$ be a unit-speed geodesic from Y to X. One has

$$g_Y(X-Y)^{1/2} \leq \int_0^L g_Y\left(\tfrac{\mathrm{d}X}{\mathrm{d}t}\right)^{1/2}\mathrm{d}t \leq C' \int_0^L (1+t)^{N/2}\mathrm{d}t \leq C''(1+L)^{N/2+1} ,$$

which gives the right part of (6).

Remark 2. The geodesic temperance requires both the temperance and (5) (or (6)). A metric can satisfy (5) without being tempered (example: $\mathrm{e}^x\mathrm{d}x^2+\mathrm{e}^{-x}\mathrm{d}\xi^2$). However, if g satisfies (5) and (3), it is tempered: the ratio g_X/g_Y is estimated by some power of the g^σ-length of the segment XY which is itself estimated, as remarked above, by a power of $g_Y^\sigma(X-Y)$.

Remark 3. There is no known example of a tempered metric g which is not geodesically tempered, but proving that a particular metric is geodesically tempered can be a challenging task. Theorem 5 below proves that this is not an issue for the Beals-Fefferman metrics.

2.2 Characterization

We shall use the classical notations

$$\operatorname{ad} B \cdot A = [B,A] = B \circ A - A \circ B$$

for the commutator of two operators. The following definition is of interest only for metrics WSE.

Definition 5. Let g be a metric WSE. The class $\widehat{S}(\lambda,g)$ is the set of functions $b \in C^\infty(\mathscr{X})$ which satisfy, for any $\boxed{k \geq 1}$,

$$\exists C_k; \quad \left|\partial_{T_1} \ldots \partial_{T_k} b(X)\right| \leq C_k \lambda(X) \text{ for } g_X(T_j) \leq 1 \,.$$

We refer to [3] for the proof of the following theorem, and just add some comments.

Theorem 2. *Let g be an admissible metric WSE which is* geodesically tempered. *Then A belongs to* $\operatorname{Op} S(1,g)$ *if and only if the iterated commutators*

$$\operatorname{ad} b_1^w \ldots \operatorname{ad} b_k^w \cdot A \,, \quad k \geq 0, \;\; b_j \in \widehat{S}(\lambda,g) \,, \tag{7}$$

are bounded on L^2.

Remark 4. It is easy to see that, for $b \in \widehat{S}(\lambda,g)$ and $a \in S(1,g)$, the Poisson bracket $\{b,a\}$ belongs to $S(1,g)$. The “only if” part of the proof comes from the fact that $b\#a - a\#b$ (a non-local integral expression whose principal part is $-\mathrm{i}\{b,a\}$) belongs also to $S(1,g)$.

It would not be sufficient to require the boundedness of the commutators with $b_j \in S(\lambda,g)$. For instance, if g is the euclidean metric of $\mathscr{X}$, the class $S(1,g)$ is nothing but $S^0_{0,0}$, and one has $\lambda = 1$. For $b_j \in S(\lambda,g) = S(1,g)$ the corresponding operators are bounded on L^2 and (7) would be valid for any A bounded on L^2. On the other hand, $\widehat{S}(\lambda,g)$ contains in that case the functions x_j and ξ_j and the theorem reduces to the criterium of Beals.

Remark 5. One can realize the importance of the geodesic temperance in this way. An essential ingredient in the proof of the “if” part is to prove a “decay of A outside the diagonal of $\mathscr{X}$”, namely that, given balls $B_Y = \left\{X \mid g_Y(X-Y) \leq r^2\right\}$ and a family, bounded in $S(1,g)$, of functions α_Y supported in B_Y, one has

$$\|\alpha_Y^w \circ A \circ \alpha_Z^w\|_{\mathscr{L}(L^2)} \leq C_N(1 + g_Y^\sigma(B_Y - B_Z))^{-N} \,,$$

where $g_Y^\sigma(B_Y - B_Z)$ means $\inf g_Y^\sigma(Y' - Z')$ for $Y' \in B_Y$ and $Z' \in B_Z$.

It turns out that, from the estimate on the commutators with $b^w \in \operatorname{Op}\widehat{S}(\lambda,g)$, one can gain only the variation of b between B_Y and B_Z. But functions in $\widehat{S}(\lambda,g)$ are Lipschitz continuous for g^σ, the variation of b cannot exceed the geodesic distance of the balls and one cannot obtain a better bound than $C_N(1+d^\sigma(B_Y,B_Z))^{-N}$ in the right hand side. The geodesic temperance asserts precisely that such a bound can compensate the ratio g_Y/g_Z.

The characterization can be extended to some cases where g is not WSE, the space $\widehat{S}(\lambda,g)$ being accordingly modified (see [3]). However, some extra conditions on g should be added; if not, [3] contains an example where the characterization fails.

Corollary 1. *Under the same assumptions on g, let M and M_1 be two g-weights. Then an operator A belongs to $\operatorname{Op}S(M,g)$ if and only if A and its iterated commutators with elements of $\operatorname{Op}\widehat{S}(\lambda,g)$ map continuously $H(M_1,g)$ into $H(M_1/M,g)$.*

Let us choose $B \in \operatorname{Op}S(M_1,g)$ having an inverse $B^{-1} \in S(M_1^{-1},g)$ (see [5, cor. 6.6.]), and another invertible operator $C \in \operatorname{Op}S(M_1/M,g)$. Then, $A' = CAB^{-1}$ satisfy the assumptions of Theorem 2, one has $A' \in \operatorname{Op}S(1,g)$ and thus $A = BA'C^{-1} \in \operatorname{Op}S(M,g)$.

Corollary 2 (Wiener property). *Assume g admissible, WSE and geodesically tempered.*
(i) *If $A \in \operatorname{Op}S(1,g)$ is invertible in $\mathscr{L}(L^2)$ then its inverse A^{-1} belongs to $\operatorname{Op}S(1,g)$.*
(ii) *If M and M_1 are two g-weights and if $A \in \operatorname{Op}S(M,g)$ is a bijection from $H(M_1,g)$ onto $H(M_1/M,g)$, then $A^{-1} \in \operatorname{Op}S(M^{-1},g)$.*

For $b_1 \in \widehat{S}(\lambda,g)$ one has $C = \operatorname{ad} b_1^w \cdot A^{-1} = -A^{-1}(\operatorname{ad} b_1^w \cdot A)A^{-1}$ which is bounded on L^2. Next

$$\begin{aligned}\operatorname{ad} b_2^w \cdot C = -(\operatorname{ad} b_2^w \cdot A^{-1})(\operatorname{ad} b_1^w \cdot A)A^{-1} - A^{-1}(\operatorname{ad} b_2^w \cdot \operatorname{ad} b_1^w \cdot A)A^{-1} \\ -A^{-1}(\operatorname{ad} b_1^w \cdot A)(\operatorname{ad} b_2^w \cdot A^{-1})\end{aligned}$$

and the three terms are bounded on L^2. By induction, one gets that all iterated commutators are bounded on L^2 and thus that A^{-1} belongs to $S(1,g)$.

Remark 6. The Wiener property is actually valid for some metrics g which are not WSE, including cases where the characterization is not valid. One has just to assume that the metric $\widetilde{g} = \lambda^{-1}g^\#$ of the paragraph 1.3 is geodesically tempered. One knows then that $A^{-1} \in \operatorname{Op}S(1,\widetilde{g})$, and proving that its symbol actually belongs to $S(1,g)$ is just a matter of symbolic calculus. The second part of the proof of [5, th. 7.6] can be reproduced as is.

2.3 Functional calculus

Given a selfadjoint operator A (bounded or unbounded) on L^2, the functional calculus associates to any Borel function f, defined on the spectrum of A, an operator $f(A)$. When f belongs to C^∞, it can be computed via the formula of Helffer-Sjöstrand:

$$f(A) = -\pi^{-1} \iint \overline{\partial}\widetilde{f}(z) R_z \,\mathrm{d}x\,\mathrm{d}y\,, \quad z = x + \mathrm{i}y\,, \tag{8}$$

where $R_z = (z-A)^{-1}$ is the resolvant, and $\widetilde{f}$ is an almost analytic extension of f.

Theorem 3. *Assume g admissible, WSE and geodesically tempered. Let $a \in S(1,g)$ be real valued and let f be a C^∞ function defined in a neighbourhood of the spectrum of a^w. Then $f(a^w)$ belongs to* $\mathrm{Op}\,S(1,g)$.

In that case, $\widetilde{f}$ can be choosen with compact support in $\mathbb{C}$ and the meaning of (8) is clear. This theorem is actually a particular case of the following one, where the assumption: A self-adjoint with domain $H(M)$, should be thought of as a condition of ellipticity. When $C^{-1}\lambda^{1/N} \le M \le C\lambda^N$, it is equivalent to $1+|a(X)| \ge C^{-1}M(X)$ and also to the existence of a parametrix $E \in \mathrm{Op}\,S(M^{-1},g)$ such that $AE-I$ and $EA-I$ belong to $\mathrm{Op}\,S(\lambda^{-\infty},g)$.

Theorem 4. *Under the same assumptions on g, let $M \ge 1$ be a g-weight. Let $a \in S(M,g)$ be real valued such that $A = a^w$ is self-adjoint with domain $H(M,g)$. Let $f \in C^\infty(\mathbb{R})$ be a "symbol of order p", i.e. such that*

$$\left|\frac{\mathrm{d}^k f(t)}{\mathrm{d}t^k}\right| \le C_k(1+|t|)^{p-k} \quad \text{for } k \ge 0\,.$$

Then $f(a^w)$ belongs to $\mathrm{Op}\,S(M^p,g)$*. Moreover, if c is the symbol of $f(a^w)$, one has $c - f\circ a \in S(M^p\lambda^{-2},g)$.*

The result is evident for $1+A^2$ and, dividing f if necessary by a power of $1+t^2$, one may assume $p<0$. One can then choose, for M large enough,

$$\widetilde{f}(x+iy) = \chi\left(\tfrac{y^2}{1+x^2}\right)\sum_{k=0}^{M} f^{(k)}(x)\frac{(\mathrm{i}y)^k}{k!}\,,$$

where $\chi \in C^\infty(\mathbb{R})$ satisfy $\chi(s) = 1$ [resp. 0] for $s \le 1/2$ [resp. $s \ge 1$]. One has

$$\left|\overline{\partial}\widetilde{f}(x+iy)\right| \le C(1+|x|)^{p-N-2}\,|y|^{N+1} \ \text{ for } N \le M-1\,, \tag{9}$$

and the integral in the right hand side of (8) is thus convergent.

For $\Im z \neq 0$, we know that R_z is a bijection of L^2 onto the domain $H(M)$ and thus, as a consequence of Corollary 2, that $R_z \in \mathrm{Op}\,S(M^{-1})$. Let us denote by r_z its Weyl symbol. We have $\|R_z\|_{\mathscr{L}(L^2)} \le 1/|\Im z|$. From the resolvant equation $R_z - R_\mathrm{i} = (\mathrm{i}-z)R_\mathrm{i}R_z$, one gets $\|R_z\|_{\mathscr{L}(L^2,H(M))} \le C(1+|z|)/|\Im z|$. The iterated commutators can be written

$$\prod(\mathrm{ad}\, b_j^w)\cdot R_z = \sum \pm R_z K_1 R_z \ldots R_z K_p R_z$$

where the sum is finite and each K_l is an iterated commutator of A with some of the b_j. Thus $\|K_l\|_{\mathscr{L}(H(M),L^2)}$ is bounded independantly of z and l, and one has

$$\left\| \prod_1^N (\operatorname{ad} b_j^w) \cdot R_z \right\|_{\mathscr{L}(L^2, H(M))} \leq C \prod_1^N \|b_j\|_{l;\widehat{S}(\lambda,g)} \frac{(1+|z|)^{N+1}}{|\Im z|^{N+1}}$$
$$\left\| \prod_1^N (\operatorname{ad} b_j^w) \cdot R_z \right\|_{\mathscr{L}(L^2)} \leq C \prod_1^N \|b_j\|_{l;\widehat{S}(\lambda,g)} \frac{(1+|z|)^{N}}{|\Im z|^{N+1}}$$

with $C = C(N)$, $l = l(N)$, denoting by $\|\cdot\|_{k,E}$ the semi-norms of a Frechet space E.

The proof of Theorem 2 shows that there exist constants C_k and N_k such that

$$\|r_z\|_{k;S(1,g)} \leq C_k \frac{(1+|z|)^{N_k}}{|\Im z|^{N_k+1}} \;, \quad \|r_z\|_{k;S(M^{-1},g)} \leq C_k \frac{(1+|z|)^{N_k+1}}{|\Im z|^{N_k+1}}$$

Let us denote by c the symbol of $f(A)$. Using the estimates above, for $k = 0$, one gets from (8) and (9)

$$\begin{aligned} |c(S)| &\leq C \iint_{|y|<(1+|x|)} \frac{(1+|x|)^N}{|y|^{N+1}} \min\left\{1\,,\, \frac{(1+|x|)}{M(S)}\right\} |y|^{N+1} (1+|x|)^{p-N-2} \, \mathrm{d}x \, \mathrm{d}y \\ &\leq C' \int (1+|x|)^{p-1} \min\left\{1\,,\, \frac{1+|x|}{M(S)}\right\} \mathrm{d}x \leq C'' M(S)^p \;. \end{aligned}$$

The estimates of the derivatives $\prod \partial_{T_j} c(S)$ for $g_S(T_j) \leq 1$ are analogous, which ends the proof.

3 Sufficient conditions for the geodesic temperance

Theorem 5. (i) *Admissible Beals-Fefferman metrics are geodesically tempered.*
(ii) *More generally, let $\mathscr{X} = \mathscr{X}_1 \oplus \mathscr{X}_2 \oplus \cdots \oplus \mathscr{X}_p$ be a decomposition of $\mathscr{X}$ as a vector space, and note $X = (X_1, \ldots, X_p)$ the components of a vector. Let g be an admissible metric such that g^σ can be written*

$$\begin{aligned} g_X^\sigma(dX) = a_1(X_1,X_2)\Gamma_1(dX_1) \,+\, & a_2(X_1,X_2)\Gamma_2(dX_2) + a_3(X_1,X_2,X_3)\Gamma_3(dX_3) \\ & + a_4(X_1,X_2,X_3,X_4)\Gamma_4(dX_4) + \cdots + a_p(X)\Gamma_p(dX_p) \;, \end{aligned}$$

where Γ_j is a positive definite quadratic form on $\mathscr{X}_j$ and a_j is a positive function of its arguments. Then g is geodesically tempered.

Remark 7. We keep the notations which are of interest for us, but (ii) could be stated for an affine space $\mathscr{X}$ on which a Riemannian metric g^σ is given. The symplectic structure and g itself play no role, the temperance reduces to

$$\left(g_Y^\sigma(\cdot)/g_X^\sigma(\cdot)\right)^{\pm 1} \leq C\left(1 + g_Y^\sigma(X-Y)\right)^N$$

and one has to prove the right part of (6).

The particular role played by X_1 and X_2 should be noted. It is only for $X_j, j \geq 3$, that a "triangular structure" of g^σ is required.

3.1 Proof of Theorem 5 (i)

We have $g_X^\sigma = \Phi(X)^2 \mathrm{d}x^2 + \varphi(X)^2 \mathrm{d}\xi^2$. The temperance and (3) can be formulated as follows:

$$(g_Y^\sigma(\cdot)/g_X^\sigma(\cdot))^{\pm 1/2} \leq C \max\{1\,;\,\Phi(Y)\,|x-y|\,;\,\varphi(Y)\,|\xi-\eta|\}^N,$$
$$1+\Phi(X)\,|x-y| + \varphi(X)\,|\xi-\eta| \leq C \max\{1\,;\,\Phi(Y)\,|x-y|\,;\,\varphi(Y)\,|\xi-\eta|\}^N. \quad (10)$$

The value of $\kappa > 0$, $\varepsilon > 0$ and $R_0 > 1$ will be fixed later, depending only of C and N above. Other constants, such as $C', C'', C_1 \dots$ may vary from line to line, but can be computed depending of C and N.

We have to prove that any curve $t \in [0,T] \mapsto X(t)$ joining a point (which we take as origin) to a point of the boundary of $\partial(B_x(R) \times B_\xi(R))$ has a length $\geq C'^{-1} R^\delta$, with $\delta > 0$ and C' independent of R. Here,

$$B_x(R) = \left\{ x \,\middle|\, |x| \leq R/\Phi(0) \right\}, \quad B_\xi(R) = \left\{ \xi \,\middle|\, |\xi| \leq R/\varphi(0) \right\}.$$

The result is evident if $R \leq R_0$: in that case, one has $g_X \geq C^{\mathrm{st}} g_0$ pour $X \in B_x(R) \times B_\xi(R)$ (with a constant depending on R_0). Thus, we will assume $R \geq R_0$. We may assume that T is the first instant when $X(t)$ reaches the boundary. Exchanging if necessary x and ξ, we may assume that $\Phi(0)\,|x(T)| = R$. Set $R' = \max_0^T \varphi(0)\,|\xi(t)|$ and let T' be the first instant when R' is reached.

We distinguish two cases.

• *Case I:* $R' \leq R^\kappa$. — Set $Y(t) = (0, \xi(t))$. By temperance, one has $\Phi(Y(t)) \geq C'^{-1} R^{-N\kappa} \Phi(0)$ and, using (3),

$$(1+\Phi(X(t))\,|x(t)|) \geq C'^{-1}(1+\Phi(Y(t))\,|x(t)|)^{1/N}.$$

Let us consider the length L of the curve between the last instant θ when $\Phi(0)\,|x(t)| = R/2$ and T. For $t \geq \theta$, one has then $\Phi(X(t))/\Phi(0) \geq C'^{-1} R^{-1+1/N-\kappa}$, and

$$L \geq \int_\theta^T \Phi(X(t))\,\left|x'(t)\right| \mathrm{d}t \geq C'^{-1} R^{-1+1/N-\kappa} \int_\theta^T \Phi(0)\,\left|x'(t)\right| \mathrm{d}t \geq C'' R^{1/N-\kappa}$$

We may now fix $\kappa = 1/(2N)$ and the result is proved, with $\delta = 1/(2N)$, in the first case.

• *Case II:* $R' \geq R^\kappa$. — We distinguish three subcases.

Subcase (IIa). — $\Phi(X)/\Phi(0) \geq R^{-1+\varepsilon}$ everywhere in $B_x(R) \times B_\xi(R')$. Then, the length of the curve is greater than

$$\int_0^T \Phi(X(t)) \left|x'(t)\right| \mathrm{d}t \geq R^{-1+\varepsilon} \int_0^T \Phi(0) \left|x'(t)\right| \mathrm{d}t \geq R^{\varepsilon} ,$$

which ends the proof, with $\delta = \varepsilon$, in this subcase.

Subcase (IIb). — $\varphi(X)/\varphi(0) \geq R'^{-1+\varepsilon}$ everywhere in $B_x(R) \times B_\xi(R')$. This is similar, the length is larger than

$$\int_0^{T'} \varphi(X(t)) \left|\xi'(t)\right| \mathrm{d}t \geq R'^{\varepsilon} \geq R^{\varepsilon\kappa} ,$$

and the theorem is proved with $\delta = \kappa\varepsilon$, in this subcase.

Subcase (IIc). — It is the remaining case and we will prove that it cannot occur provided that ε and R_0 be conveniently choosen. There should exist $Y_1 = (x_1, \xi_1)$ and $Y_2 = (x_2, \xi_2)$ in $B_x(R) \times B_\xi(R')$ such that

$$\Phi(Y_1)/\Phi(0) \leq R^{-1+\varepsilon} \quad \text{and} \quad \varphi(Y_2)/\varphi(0) \leq R'^{-1+\varepsilon} .$$

Let us consider the point $Z = (x_2, \xi_1)$. One has $\Phi(Y_1) \left|x_2 - x_1\right| \leq 2R^{\varepsilon}$ and thus

$$\Phi(Z) \leq C' R^{N\varepsilon} \Phi(Y_1) \leq C' R^{-1+(N+1)\varepsilon} \Phi(0) . \tag{11}$$

Then, assuming $(N+1)\varepsilon < 1$,

$$\Phi(Z) \left|x_2\right| \leq C' \left(\tfrac{\Phi(0)\left|x_2\right|}{R}\right)^{1-(N+1)\varepsilon} \left(\Phi(0) \left|x_2\right|\right)^{(N+1)\varepsilon} \leq C' \left(\Phi(0) \left|x_2\right|\right)^{(N+1)\varepsilon} .$$

The same computation, where R is replaced by R', shows that

$$\varphi(Z) \left|\xi_1\right| \leq C' \left(\varphi(0) \left|\xi_1\right|\right)^{(N+1)\varepsilon} .$$

Applying (10) between 0 and Z, we get

$$\begin{aligned}(1 + \Phi(0) \left|x_2\right| + \varphi(0) \left|\xi_1\right|) &\leq C(1 + \Phi(Z) \left|x_2\right| + \varphi(Z) \left|\xi_1\right|)^N \\ &\leq C'(1 + \Phi(0) \left|x_2\right| + \varphi(0) \left|\xi_1\right|)^{N(N+1)\varepsilon} .\end{aligned}$$

Now, fix $\varepsilon = \frac{1}{2N(N+1)}$. The inequality imply the existence of a constant C_1 such that $(1 + \Phi(0) \left|x_2\right| + \varphi(0) \left|\xi_1\right|) \leq C_1$. By temperance, one has $\Phi(0)/\Phi(Z) \leq C_2$, which is to compare with (11). One gets

$$R^{1-(N+1)\varepsilon} \leq C' \frac{\Phi(0)}{\Phi(Z)} \leq C' C_2 ,$$

which is impossible for $R \geq R_0$ if we choose, for instance, $R_0 = 2(C'C_2)^2$.

3.2 Proof of Theorem 5 (ii)

The proof of part (i) is also the proof of the case $p=2$ (and also of course $p=1$). As remarked above, the symplectic structure plays no role. Thus, R^n_x, $\mathbb{R}^n_\xi$ and their canonical quadratic forms can be replaced by $\mathscr{X}_1$, $\mathscr{X}_2$, Γ_1 and Γ_2.

For $p>2$, the theorem is a consequence, by induction, of the following lemma.

Lemma 1. *Assume $\mathscr{X}=\mathscr{Y}\oplus\mathscr{Z}$ and let G^σ be a Riemannian metric on $\mathscr{X}$ of the following form*

$$G^\sigma(X,\mathrm{d}X)=g^\sigma(Y,\mathrm{d}Y)+a(Y,Z)\Gamma(\mathrm{d}Z),$$

where g^σ is a Riemannian metric on $\mathscr{Y}$, a is a positive function on $\mathscr{X}$ and Γ is a positive definite quadratic form on $\mathscr{Z}$. Assume the temperance of G^σ and the geodesic temperance of g^σ. Then the geodesic temperance is valid for G^σ.

If we denote by d^σ the geodesic distance for g^σ on $\mathscr{Y}$ and by D^σ the geodesic distance for G^σ on $\mathscr{X}$, there exists thus constants C and N such that

$$\left(G^\sigma_{X_1}/G^\sigma_{X_2}\right)^{\pm 1}\le C\left(1+G^\sigma_{X_1}(X_2-X_1)\right)^N\ , \tag{12}$$

$$\left(g^\sigma_{Y_1}/g^\sigma_{Y_2}\right)^{\pm 1}\le C\left(1+g^\sigma_{Y_1}(Y_2-Y_1)\right)^N\ , \tag{13}$$

$$C^{-1}\left(1+d^\sigma(Y_1,Y_2)\right)^{1/N}\le\left(1+g^\sigma_Y(Y_1-Y_2)\right)\le C\left(1+d^\sigma(Y_1,Y_2)\right)^N\ . \tag{14}$$

Let us consider two points X_0 and X_1 and a curve $[0,1]\ni t\mapsto X(t)=(Y(t),Z(t))$ joining these two points. Let us denote by L the G^σ-length of this curve, and set $R^2=G^\sigma_{X_0}(X_1-X_0)$. We want to prove that there exist C' and $\delta>0$, depending just of C and N above, such that $L\ge C'^{-1}R^\delta$. We will assume, as we may, that $R\ge 1$. The value of κ, $0<\kappa<1$, will be fixed later and we distinguish two cases.

• *Case I:* $\forall t,\ g^\sigma_{Y_0}(Y(t)-Y_0)^{1/2}\le R^\kappa/2$. — One has then $(a(X_0)\Gamma(Z_1-Z_0))^{1/2}\ge R/2$. Let us consider the curve $t\mapsto P(t)=(Y_0,Z(t))$. We can apply the case $p=1$ of the theorem to the metric $a(Y_0,Z)\Gamma(\mathrm{d}Z)$ on the affine space $\{Y_0\}\times\mathscr{Z}$. These metrics depend on Y_0, but they are tempered with the same constants C and N, and thus they are geodesically tempered with uniform constants. One has thus

$$\int_0^1\left(a(P(t))\Gamma(\dot Z(t))\right)^{1/2}\mathrm{d}t\ge C'^{-1}R^\alpha$$

with $\alpha>0$ and C' depending just of C and N.

The temperance of G^σ imply $a(X(t))^{1/2}\ge C'^{-1}R^{-N\kappa}a(P(t))^{1/2}$ and thus

$$\begin{aligned}L\ge\int_0^1\left(a(X(t))\Gamma(\dot Z(t))\right)^{1/2}\mathrm{d}t&\ge C'^{-1}R^{-N\kappa}\int_0^1\left(a(P(t))\Gamma(\dot Z(t))\right)^{1/2}\mathrm{d}t\\&\ge C''^{-1}R^{\alpha-N\kappa}\ .\end{aligned}$$

Fix now $\kappa=\alpha/(2N)$ and the lemma is proved with $\delta=\alpha/2$ in this first case.

• *Case II:* $\exists t_0,\ g^\sigma_{Y_0}(Y(t_0)-Y_0)^{1/2} \geq R^\kappa/2$. — Let us consider the curve $[0,t_0] \ni t \mapsto Y(t)$ in $\mathscr{Y}$. By (14), one gets

$$L \geq \int_0^{t_0} \left(g^\sigma_{Y(t)}(\dot{Y}(t))\right)^{1/2} \mathrm{d}t \geq C^{-1}(1+g^\sigma_{Y_0}(Y(t_0)-Y_0))^{1/N} \geq C'^{-1}R^{2\kappa/N}\ ,$$

which ends the proof of the lemma and of Theorem 5.

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