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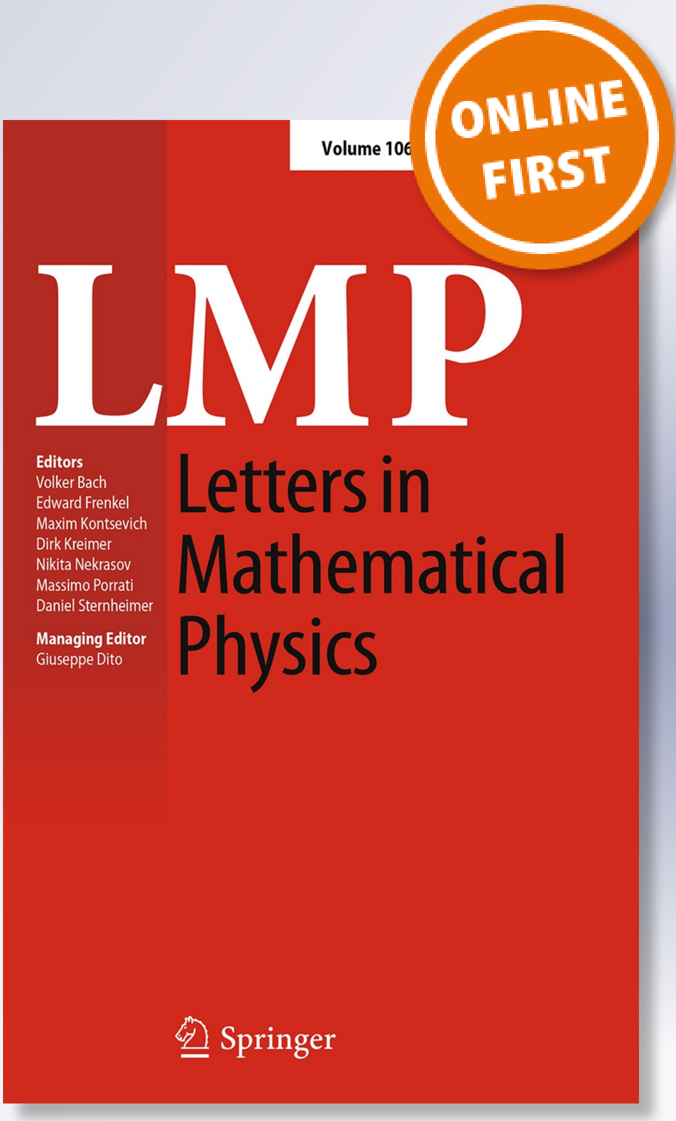

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A Tribute to Louis Boutet de Monvel

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Abstract. Louis Boutet de Monvel, an eminent mathematician of our time, passed away on December 25, 2014. This article addresses his outstanding personality and his great contributions to mathematics.

Mathematics Subject Classification. 32-XX, 35-XX, 58-XX.

Keywords. pseudodifferential operators, boundary value problems, index formula, double characteristics, Hermite operators, Bergman and Szegö projectors, Toeplitz operators, D-modules, relative and equivariant index, star-product.

When to the sessions of sweet silent thought
I summon up remembrance of things past,
I sigh the lack of many a thing I sought,
And with old woes new wail my dear time's waste:
Then can I drown an eye, unused to flow,
For precious friends hid in death's dateless night
W. SHAKESPEARE

Louis Boutet de Monvel passed away on December 25, 2014. Like many others, I lost a friend, and the mathematical community lost an eminent and very influential mathematician.

His work covers mainly three subjects usually considered as different (at least by the AMS classification): partial differential equations, several complex variables and global analysis on manifolds. However, one can also say that his entire work tends to make these three subjects just but one. Actually, his domain of interest was much broader, from algebraic geometry and topology to all aspects of mathematical physics.

Louis entered the École Normale Supérieure in 1960, where I met him the following year. It is there that he discovered, as I will explain, what will be the main stream of his research. He had been professor in Alger, Nice, Grenoble and mainly in Paris. In particular, he directed the Centre of Mathematics of the École Normale Supérieure for 8 years.

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Let me go back to the academic year 1963–1964, when Louis was still a student at the ENS. That year, the Cartan seminar was organized jointly by Henri Cartan and Laurent Schwartz, and it was entirely devoted to the proof of the Atiyah–Singer index theorem. Many participants were able to master either the topological member or the analytic member of the equality, but few were able to do both. Louis, who was the youngest of the lecturers, belonged to the latter category.

During the whole seminar, there had been some kind of suspense about the sign in the index formula for odd dimensional manifolds. It is Louis who, in the last talk, making a thorough study of the one-dimensional case, was able to remove the indeterminacy. Global analysis and index formulas will remain a constant concern for him: he gave such formulas, absolute or relative or equivariant, for boundary value problems, Toeplitz operators and $\mathcal{D}$-modules.

After the seminar, an extra talk was given by M. Atiyah, sketching an index formula for differential boundary problems. Louis found there the subject of his thesis, which led him to his famous theory of pseudodifferential boundary value problems.

1. Boundary Value Problems

Louis devoted several papers to the theory of pseudodifferential boundary problems [1,2,4] converging toward his famous paper [5] in *Acta Mathematica* where he defines the full symbolic calculus of these operators—nowadays usually called "the Boutet de Monvel calculus"—and apply this to the proof of the index formula for general elliptic boundary value problems.

1.1. THE ALGEBRA OF "GENERAL GREEN OPERATORS"

Given a C^∞ manifold with boundary $\overline{\Omega}$ (let us say compact for simplicity) and vector bundles E,E' on $\overline{\Omega}$ and F, F' on $\partial\Omega$, he defines a family of operators containing classical boundary value problems as well as their parametrices and which is moreover stable by composition. General Green operators are "matrices":

$$A = \begin{pmatrix} P+G & K \\ T & Q \end{pmatrix} \quad : \quad \begin{matrix} C^\infty(\overline{\Omega}, E) \\ \oplus \\ C^\infty(\partial\Omega, F) \end{matrix} \quad \longrightarrow \quad \begin{matrix} C^\infty(\overline{\Omega}, E') \\ \oplus \\ C^\infty(\partial\Omega, F') \end{matrix} \quad . \tag{1}$$

(a) P stands for $f \mapsto (P\widetilde{f})|_{\overline{\Omega}}$ where P is a pseudodifferential operator defined in a neighbourhood of $\overline{\Omega}$ having the *transmission property*, applied to the extension $\widetilde{f}$ of f by 0 outside $\overline{\Omega}$.
(b) K is a *Poisson operator* (see more details below).
(c) T is a *trace operator*. These operators contain classical ones (pseudodifferential operators on $\partial\Omega$ applied to traces of derivatives of f) as well as the adjoints of Poisson operators.
(d) Q is a pseudodifferential operator on $\partial\Omega$.

(e) G is a *singular Green operator*. These operators (for instance, products KT of Poisson and trace operators) map $\mathcal{D}'(\overline{\Omega})$ into functions which are C^∞ in Ω, but not in $\overline{\Omega}$.

Already analyzed in [1,2] the *transmission property* ensures that, for $f \in C^\infty(\overline{\Omega})$, one has $(P\widetilde{f})|_{\overline{\Omega}} \in C^\infty(\overline{\Omega})$. In the simplest case where the symbol p of P has an asymptotic expansion $p(x,\xi) \sim \sum_k p_k(x,\xi)$ with p_k homogeneous in ξ of degree d_k, the condition reads as follows:[1] $p_k(x,-\xi) - e^{i n d_k} p_k(x,\xi)$ vanishes to infinite order on $\{x_n = 0, \xi' = 0\}$. That condition is always satisfied by differential operators and by their parametrices.

1.2. THE SYMBOLIC CALCULUS

The symbol of A in (1) have two parts: the interior symbol, which is just the usual symbol of the pseudodifferential operator P defined on the cotangent bundle $T^\star\Omega$, and the boundary symbol, which is a *Wiener–Hopf operator* depending on $(x',\xi') \in T^\star\partial\Omega$.

Wiener–Hopf operators are also matrices of operators, which correspond to the boundary calculus in dimension 1, seen in the Fourier analysis. They act on H^+ which is the space of Fourier–Laplace transforms of elements of $\mathcal{S}(\overline{\mathbb{R}^+})$ (functions vanishing for $x<0$, rapidly decreasing at $+\infty$, and C^∞ up to the origin for $x \geq 0$). Equivalently, elements of H^+ are C^∞ functions on the real line with a regular pole at infinity, vanishing at infinity, and having an analytic extension in the lower half-plane. Wiener–Hopf operators are matrices

$$\begin{pmatrix} p+g & k \\ t & q \end{pmatrix} \quad : \quad \begin{matrix} H^+ \otimes E \\ \oplus \\ F \end{matrix} \longrightarrow \begin{matrix} H^+ \otimes E' \\ \oplus \\ F' \end{matrix}, \tag{2}$$

where E, E', F, F' are finite-dimensional vector spaces.

Let us just describe the Poisson term k. It is associated with an element $k \in H^+ \otimes \mathcal{L}(F,E')$ and defined by $F \ni u \mapsto k \cdot u \in H^+ \otimes E'$.

Now, let $k(x',\xi)$ be a C^∞ function on $\mathbb{R}^{n-1} \times \mathbb{R}^n$ having the following expansion:[2]

$$k(x',\xi) = \sum a_p(x',\xi') \frac{(\langle \xi \rangle - i\xi_n)^p}{(\langle \xi \rangle + i\xi_n)^{p+1}},$$

where (a_p) is a rapidly decreasing sequence in $S^d_{1,0}$. It can be considered as an element of $H^+_{\xi_n}$ depending on (x',ξ'). The Poisson operator of degree d and symbol k is given by

$$Kf(x) = (2\pi)^{-n} \int e^{ix_n\xi_n}\, \mathrm{d}\xi_n \int_{\mathbb{R}^{n-1}} e^{ix'\cdot\xi'} k(x',\xi',\xi_n) \widehat{f}(\xi')\, \mathrm{d}\xi'.$$

[1] Explicit formulas will be given for $\Omega = \overline{\mathbb{R}^{n+}} = \{(x',x_n) \mid x_n \geq 0\}$, the bundles being trivialized (or omitted).

[2] Such expansions in H^+ correspond to series of Laguerre functions in $\mathcal{S}(\overline{\mathbb{R}^+})$.

Other terms in (1) and (2) have analogous definitions. The Wiener–Hopf operators form an algebra, and the principal boundary symbol of general Green operators behaves as expected with respect to composition and change of variables.

1.3. THE INDEX FORMULA

General Green operators acts between Sobolev spaces of convenient order; elliptic ones are Fredholm and thus have a finite index. Louis associates to any such operator A (or rather to its principal symbol) a virtual bundle $[A]\in K(T^\star\Omega)$ and he proves that the index of A is equal to the topological index of $[A]$. The scope of that formula is much more general than the case of classical differential boundary value problems studied by M. Atiyah and R. Bott.

In my opinion, the proof given by Louis is the most natural and the most straightforward. He uses of course the invariance of the index by deformation and compact perturbations, but can take full advantage of his general symbolic calculus. Using the composition of an arbitrary elliptic general Green operator with special ones for which he knows that their index is 0, he can reduce the problem to diagonalized operators $\left(\begin{smallmatrix} P & 0\\ 0 & Q\end{smallmatrix}\right)$ with $P=$ identity near $\partial\Omega$, for which the classical Atiyah–Singer formula gives immediately the result.

2. Analytic Pseudodifferential Calculus

With Krée [3], Louis developed the theory of pseudodifferential operators whose symbol belongs to the Gevrey class G^σ, $\sigma\geq 1$. This includes the important case ($\sigma=1$) of the analytic calculus. He generalized this to the boundary calculus of analytic singular Green operators in [4].

More precisely, (formal) symbols of class σ and order r are defined as formal sums $\sum p_k(x,\xi)$, where p_k is homogeneous in ξ of degree $r-k$, satisfying the following estimates on each compact set:

$$\left|\partial_x^\alpha\partial_\xi^\beta p_k(x,\xi)\right|\leq c\,A^{k+|\alpha+\beta|}\,|\xi|^{r-k-|\beta|}\left((k+|\alpha|)!\right)^\sigma\beta! \tag{3}$$

The results are as expected: 1. For any formal symbol, there exists an associated pseudodifferential operator, unique up to σ-regularizing ones. 2. Such an operator P maps ultradistributions $\in G^{\sigma\prime}$ with compact support [analytic functionals for $\sigma=1$] into ultradistributions [hyperfunctions for $\sigma=1$] and Pu is Gevrey in any open set where u is 3. The usual formulas of symbolic calculus for composition, adjoint and parametrices (of elliptic symbols) are valid.

Estimates like (3) could have led to monstrous computations, but the authors introduced a very elegant tool: the *formal norm* of a symbol, whose finiteness is equivalent to (3), which is the formal series:

$$N_\sigma(p,T)=\sum_{\alpha,\beta,k}\left(\frac{2(2n)^{-k}k!}{(k+|\alpha|)!^\sigma(k+|\beta|)!}\right)\left|\partial_x^\alpha\partial_\xi^\beta p_k\right|T^{2k+|\alpha+\beta|}.$$

One has $N_\sigma(p\#q,T) \ll N_\sigma(p,T)\cdot N_\sigma(q,T)$, where $\ll$ means that the difference of the series has nonnegative coefficients. This is a key tool for composition and invertibility and, a few years later, M. Sato, T. Kawai and M. Kashiwara had to call on these formal norms when they developed their famous theory.

3. Operators with Double Characteristics

The study of pseudodifferential operators with double characteristics and of their hypoellipticity is a subject which is important in itself. Moreover, it has deep links with complex analysis, the system of boundary Cauchy–Riemann equations entering in that category. The analysis of Louis led him, a few years later, to the Toeplitz calculus.

3.1. HYPOELLIPTICITY

Such a pseudodifferential operator P, of order m, has a symbol

$$\sigma(P) = p(x,\xi) = p_m(x,\xi) + p_{m-1}(x,\xi) + \cdots, \tag{4}$$

such that p_m vanishes of order 2 on a conic submanifold Σ of the phase space. It is elliptic outside Σ and the problem is to give conditions which guarantee that it is hypoelliptic "with loss of one derivative", i.e.

$$Pu \in H^s \Longrightarrow u \in H^{s+m-1} \text{ locally },$$

where H^s is the usual Sobolev space.

Joint works with Treves [6,7] study that problem when Σ is symplectic. In [8], Louis develops a symbolic calculus allowing to construct parametrices of P when Σ is involutive or symplectic and give applications to boundary Cauchy–Riemann equations. More general submanifolds Σ, as well as multiplicities larger than two, are studied in a joint work with Grigis and Helffer [10].

Let us describe the result when Σ is symplectic of codimension 2ν and when p_m vanishes of order exactly 2 on Σ (i.e. P is transversally elliptic). For each $(x,\xi)\in\Sigma$, important invariants are: a convex angle $\Gamma_{x,\xi}$; the set of values of the transversal Hessian of p_m at that point; the eigenvalues $\pm i\lambda_j$, $j=1,\dots,\nu$ of the operator on the normal space to Σ associated via the symplectic form to that Hessian; the value $p_{m-1}(x,\xi)$ at that point. An important invariant on Σ is

$$I_2(P) = p_{m-1} - \tfrac{1}{2i}\sum \frac{\partial^2 p_m}{\partial\xi_k \partial x_k} + \sum_1^\nu \lambda_j.$$

Then, P is hypoelliptic with loss of 1 derivative and has a parametrix as described below if and only if one has:

$$\forall \alpha \in \mathbb{N}^\nu, \quad \sum_j \alpha_j\lambda_j + I_2(P) \neq 0. \tag{5}$$

3.2. NEW SYMBOLIC CALCULUS

In the usual classes of pseudodifferential operators, one can just hope to find parametrices belonging to the class $\operatorname{Op} S^{\cdot}_{1/2,1/2}$ which has no symbolic calculus. Louis constructed in [8] a two-stage more refined calculus which allows to handle this case as well as many other problems.

The first calculus introduces classes $S^{m,k}(\Sigma)$ defined (in a simplified case) as the symbols $p = p_m + p_{m-1/2} + p_{m-1} + \cdots$, such that

$$p_{m-j} \text{ vanishes of order } k - 2j \text{ on } \Sigma \text{ for } 2j < k.$$

The symbolic calculus for $\operatorname{Op} S^{m,k}$ works as can be expected for composition, adjoint, change of variables and transformation by elliptic Fourier integral operators.

Now, going back to the operator P defined by (4), one has $P \in S^{m,2}$ and, thanks to the condition $I_2(P) \neq 0$, the symbolic calculus allows to find a first approximate inverse $Q_1 \in S^{-m,-2}$ such that

$$\begin{aligned} &PQ_1 = I + R_1 \quad \text{with } R_1 \in \operatorname{Op} \mathcal{H}^0 \\ &\text{where} \quad \mathcal{H}^m(\Sigma) = \bigcap_{N>0} S^{m-N/2,-N}(\Sigma). \end{aligned} \tag{6}$$

Moreover, using composition with Fourier integral operators, one can assume that $\mathbb{R}^n = \mathbb{R}^{n-\nu} \times \mathbb{R}^{\nu}$, $x = (y,t)$, $\xi = (\eta, \tau)$ and that $\Sigma = T^{\star}Y$ is defined by $t = \tau = 0$.

The elements of $\operatorname{Op} \mathcal{H}^0$ are infinitely regularizing outside Σ, but they do not regularize near Σ and Q_1 is not a good parametrix. The next step is a decomposition into Hermite operators.

3.3. HERMITE OPERATORS

In the model situation, the Hermite operators of degree m are defined by

$$Hf(y,t) = (2\pi)^{n-\nu} \int e^{iy\cdot\eta} h(y,t,\eta) \widehat{f}(\eta)\, \mathrm{d}\eta, \tag{7}$$

where h belongs to the class $\mathcal{H}^{m+\nu/4}$ relatively to $t = 0$ (and in particular has a fast decay in $(1 + |\eta|^{1/2}|t|)^{-N}$ for all N).

There is a good symbolic calculus relating Hermite operators and pseudodifferential ones. Moreover, any $R \in \operatorname{Op} \mathcal{H}^m(\Sigma)$ can be written (modulo infinitely regularizing operators)

$$R = \sum_{\alpha,\beta} H_\alpha Q_{\alpha\beta} H^{\star}_{\beta}, \tag{8}$$

where $(Q_{\alpha\beta})$ is a rapidly decreasing sequence of pseudodifferential operators of degree m on Y and where the H_α are the elementary Hermite operators, corresponding to $h = |\eta|^{\nu/4} h_\alpha(|\eta|^{1/2} t)$ in (7), the h_α being the classical Hermite functions of ν variables.

Going back to (6), what we expect from the Hermite calculus is to solve

$$PQ_2 = R_1 + R_2 \text{ with } Q_2 \in \mathcal{H}^{-m+1} \quad \text{and } R_2 \in \mathcal{H}^{-1/2}, \tag{9}$$

and now R_2 will be regularizing and $Q_1 - Q_2$ will be a good right parametrix.

The symbolic calculus shows that $PH_\alpha \sim \sum H_\beta Q_{\alpha\beta}$, where $(Q_{\alpha\beta})$ is a triangular matrix of pseudodifferential operators on Y whose diagonal terms have symbols precisely equal to the values $\alpha \cdot \lambda + I_2(P)$ which are not zero by (5). In view of (8), inverting that matrix allows to solve (9) and thus to get a parametrix and prove hypoellipticity.

3.4. BOUNDARY CAUCHY–RIEMANN OPERATORS

Let X be a real hypersurface of $\mathbb{C}^n$, one can define the $\overline{\partial}_b$ complex, acting on restrictions of differential forms of type $(0,*)$, the Kohn Laplacian $\square_b = \overline{\partial}_b \overline{\partial}_b^\star + \overline{\partial}_b^\star \overline{\partial}_b$ and, for each $(x,\xi) \in T^\star X$, the Levi form.

The characteristic set of $\square_b$, which is also the set of (x,ξ) for which the sequence of symbols of $\overline{\partial}_b$ is not exact, is a submanifold of $T^\star X$ of codimension $2n-2$. It is symplectic assuming that the Levi form is nondegenerate. Let $(q, n-1-q)$ be its signature at $(x,\xi) \in \Sigma$. Using the previous results of hypoellipticity, Louis shows in [8] that microlocally, $\overline{\partial}_b$ has no cohomology modulo C^∞, except in dimension q.

In [9], he gave a theorem of *global embeddability of abstract CR-manifold*. Such a manifold X of dimension $2n-1$ is given with a subbundle T'' of the complexified tangent bundle TX of dimension $n-1$, such that T'' and $\overline{T''}$ are linearly disjoint. Moreover, T'' is formally integrable which means that the bracket of two sections of T'' is still a section of T''. One can say that $f \in C^\infty(X)$ is holomorphic if one has $\langle \overline{Z}, df \rangle = 0$ for any section $\overline{Z}$ of T''. One can also define the Levi form which is a hermitian quadratic form.

The theorem asserts that if X is compact, if $2n-1 \geq 5$ and if the Levi form is positive definite (or negative definite), then there exists a complex manifold whose X is the C^∞ boundary: $\widetilde{Y} = Y \cup X$, of complex dimension n, such that T'' is the bundle of antiholomorphic vectors tangent to X. The restriction on the dimension is essential, as shown, for $2n-1=3$, by a celebrated counter-example of L. Nirenberg. The compactness assumption, which allows to use the Hodge theory, is also crucial.

In the abstract situation, it is possible to construct a complex $\overline{\partial}_b$ and a Kohn Laplacian $\square_b$ to which the same arguments as above can be applied. In particular, except for $2n-1=3$, $\square_b$ in degree 1 is hypoelliptic. This allows to find, near each point, n independant holomorphic functions defining a local immersion of X in $\mathbb{C}^n$. Glued together, these immersions define the manifold Y.

4. Bergman and Szegö Kernels

In [11], Louis and Johannes Sjöstrand gave a very precise description of the Schwartz kernels of the Bergman and Szegö projectors for a strictly pseudoconvex open set of $\mathbb{R}^n$. This extends the analysis of C. Fefferman of the restrictions of these kernels to the diagonal. The proof uses the theory of Fourier integral operators with complex phase and that of operators with double characteristics.

Let Ω be a strictly pseudoconvex bounded open set of $\mathbb{C}^n$ with C^∞ boundary $X=\partial\Omega$. The Bergman projector B is the orthogonal projector, in $L^2(\Omega)$, on the space of holomorphic functions belonging to L^2. The Szegö projector S is the orthogonal projector, in $L^2(X)$, on the Hardy space $H^2(X)=\ker\overline{\partial}_b\cap L^2(X)$, the space of traces on X of holomorphic functions.

There is a strictly pseudoconvex function ρ such that Ω is defined by $\rho>0$. This allows to define a phase function $\Psi\in C^\infty(\mathbb{C}^n\times\mathbb{C}^n)$ such that

$$\Psi(x,x)=\tfrac{1}{i}\rho(x),\qquad \Psi(x,y)=-\overline{\Psi(y,x)},$$
$$\overline{\partial}_x\Psi \text{ and } \partial_y\Psi \text{ vanish of infinite order for } y=x.$$

The main theorem asserts that there exists $F,G\in C^\infty(X\times X)$ and $F',G'\in C^\infty(\overline{\Omega}\times\overline{\Omega})$ such that, denoting by $B(\cdot,\cdot)$ and $S(\cdot,\cdot)$ the corresponding kernels, one has

$$\begin{aligned} S&=F(-i\Psi)^{-n}+G\log(-i\Psi)\\ B&=F'(-i\Psi)^{-n-1}+G'\log(-i\Psi). \end{aligned}\tag{10}$$

More precisely, there exist classical symbols $s(x,y,t)\sim\sum t^{n-k-1}s_k(x,y)$ of order $n-1$ on $X\times X\times\mathbb{R}^+$ and $b(x,y,t)\sim\sum t^{n-k}b_k(x,y)$ of order n on $\overline{\Omega}\times\overline{\Omega}\times\mathbb{R}^+$, such that

$$\begin{aligned} S(x,y)&=\int_0^\infty e^{it\Psi(x,y)}s(x,y,t)\,\mathrm{d}t\\ B(x,y)&=\int_0^\infty e^{it\Psi(x,y)}b(x,y,t)\,\mathrm{d}t, \end{aligned}$$

modulo C^∞ functions.

4.1. THE MICROLOCAL MODEL

A microlocal model of the Szegö projector plays a crucial role in the proof and it will be also very important in the theory of Toeplitz operators. Let H_0 be the operator from $L^2(\mathbb{R}^p_x)$ into $L^2(\mathbb{R}^{p+q}_{x,y})$ defined by

$$H_0 f(x,y)=(2\pi)^{-p}\int_{\mathbb{R}^p} e^{ix\cdot\xi-|y|^2|\xi|/2}\left(\tfrac{|\xi|}{\pi}\right)^{q/4}\widehat{f}(\xi)\,\mathrm{d}\xi.\tag{11}$$

It is a Fourier integral operator with complex phase, and it is also a Hermite operator. It is an isometry from $L^2(\mathbb{R}^p)$ onto the subspace of $L^2(\mathbb{R}^{p+q})$ made of solutions of $\frac{\partial}{\partial y_j}+y_j|D_x|$. The orthogonal projector in $L^2(\mathbb{R}^p)$ on the traces of these functions is thus $S_0=H_0^\star H_0$.

It turns out that, for $p = n$, $q = n-1$, the actual Szegö projector S can be conjugated microlocally to S_0 by elliptic Fourier integral operators.

4.2. THE LOGARITHMIC TERM

The logarithmic term of the Bergman kernel is not the leading term of the expansion, but it is nevertheless a biholomorphic local invariant.

Taking advantage of the results of Kashiwara and Fefferman, Louis came back to the analysis of the singularities of the Bergman kernel in [16,17]. In particular, he proved that, in dimension 2, if the coefficient of the logarithmic term (G' in (10)) vanishes at order 2 on $\partial\Omega$ in a neighbourhood of a point of $\partial\Omega$, then $\partial\Omega$ is locally biholomorphic to a sphere.

A somewhat related result is his analysis [27,28] of the *logarithmic trace* of K. Hirachi, where he proved that, for a Szegö projector, this trace is always 0, and thus cannot provide any new CR or contact invariant.

5. Toeplitz Operators

These operators have become a major tool in complex analysis and in the study of contact structures. Louis defines these operators in several complex variables [12]. He develops their symbolic calculus and proves an index formula. His book [13] with Victor Guillemin extends the concept up to defining a quantization of general contact manifolds and proves refined results on the spectrum of these operators.

5.1. SEVEREAL COMPLEX VARIABLES

Let Ω be a strictly pseudoconvex bounded open subset of $\mathbb{C}^n$ (or more generally a complex analytic space with singularities, but smooth near $X = \partial\Omega$). Let Σ^+ be the half-line bundle over X (which has to be considered as a conic symplectic manifold) made of points of T^*X which are characteristic for $\overline{\partial}_b$ and where $\overline{\partial}_b$ is not hypoelliptic. Let $\mathcal{O}^s = H^s(X) \cap \ker \overline{\partial}_b$ be the space of elements of the Sobolev space $H^s(X)$ which extends as holomorphic functions in Ω.

A Toeplitz operator of degree m is an operator which can be factorized: $T_Q = SQ$, where S is the Szegö projector and Q is a pseudodifferential operator of degree m on X. It maps $\mathcal{O}^s$ into $\mathcal{O}^{s-m}$.

The important result is that the Toeplitz operators form an algebra of pseudolocal operators. This algebra is locally (and modulo regularizing operators) isomorphic to the algebra of pseudodifferential operators in n real variables. The principal symbol of T_Q is a C^∞ function on Σ^+, homogeneous of degree m, defined by $\sigma_m(T_Q) = \sigma_m(Q)|_{\Sigma^+}$. There is a symbolic calculus analogous to the pseudodifferential one.

In particular, a Toeplitz operator (or system) whose symbol is invertible is said to be elliptic. It has then a parametrix and thus a finite index in the spaces $\mathcal{O}^s$. Louis gave in [12] an index formula, analogous to the Atiyah–Singer's one.

More recently, in a joint paper with Leichtnam et al. [31], he develops an equivariant Toeplitz calculus and defines the asymptotic equivariant index of Toeplitz operators. This is used for giving a new proof of the Atiyah–Weinstein conjecture on the index of Fourier integral operators and the relative index of CR structures.

5.2. QUANTIZATION OF CONTACT MANIFOLDS

For the results above, the microlocal model of the Szegö projector (11) plays a crucial role. It permits to define in [13] generalized Toeplitz operators associated with any oriented contact manifold or, which is equivalent, any conic symplectic manifold.

Let X be a compact manifold and Σ be a closed conic symplectic submanifold of T^*X. A *Toeplitz structure* on Σ is defined by a projector π_Σ in $L^2(X)$, with image H_Σ, which is microlocally equivalent to (11), via Fourier integral operators. A Toeplitz operator $T: C^\infty \cap H_\Sigma \to C^\infty \cap H_\Sigma$ is then defined as a product $\pi_\Sigma \circ Q$ with Q pseudodifferential. Its principal symbol is $\sigma_Q|_\Sigma$ and all the results above extend to this general situation.

5.3. SPECTRAL THEORY

If T is a self-adjoint Toeplitz operator (of order, say, 1) whose symbol is positive, then T has a discrete spectrum $\lambda_1 \leq \lambda_2 \leq \lambda_3 \cdots \to +\infty$. Let E be the generating function:

$$E(t) = \mathrm{Tr}(e^{itT}) = \sum e^{i\lambda_j t}.$$

The main result of Louis and Victor, proved using the Hermite calculus, is the following trace formula, relating the spectrum to the periodic bicharacteristics γ of the Hamiltonian T:

$$E(t) \cong \sum_\gamma C_\gamma (t - \tau_\gamma + i0)^{-1} \quad (\mathrm{mod}\ L^1_{\mathrm{loc}}(\mathbb{R}^+)), \tag{12}$$

where the sum is extended to all periodic trajectories, which are assumed to be nondegenerate. Here, τ_γ is the period of γ and the constant C_γ involves the primitive period, the Poincaré map and the integral over γ of the subprincipal symbol.

As consequences, many theorems known in the pseudodifferential setting are extended to Toeplitz operators. The counting function satisfies a Weyl law:

$$\#\{j \mid \lambda_j < \lambda\} = \frac{\mathrm{vol}\{p \in \Sigma \mid \sigma_T(p) \leq 1\}}{(2\pi)^\nu} \lambda^\nu + \mathrm{O}(\lambda^{\nu-1}), \tag{13}$$

where $\dim \Sigma = 2\nu$ and vol is the symplectic volume.

When all trajectories are periodic with the same period τ, and assuming that the integral of the subprincipal symbol over any such γ is independent of γ, they obtain the equivalent of the results of Colin de Verdière in the pseudodifferential framework: the eigenvalues are contained into the union of intervals I_m centered at $2\pi m/\tau$ and of size $\sim m^{-1}$ and there is a polynomial P of degree $\nu - 1$ such that the number of eigenvalues in I_m is equal to $P(m)$ for large m. Moreover, they give a topological formula allowing to compute P. Another result is the definition of a *Hilbert polynomial*, having a topological and an analytic interpretation, attached to any compact symplectic manifold M as long as the cohomology class of the two form in $H^2(M, \mathbb{R})$ belongs actually to $H^2(M, \mathbb{Z})$.

6. Relative Index Theorem

6.1. ALMOST ELLIPTIC SYSTEMS

For differential systems, Louis defines in [15] an extension of the notion of ellipticity. If X is a compact real manifold, one can consider small tubular neighbourhoods X_ϵ of X in its complexification, which are strictly pseudoconvex for ϵ small. Given a complex D of differential operators with analytic coefficients on X, they extend to X_ϵ and thus define a complex D_ϵ of Toeplitz operators on ∂X_ϵ. One says that D is *almost elliptic* if D_ϵ is elliptic for ϵ small.

Elliptic systems of course, but also holonomic ones, are almost elliptic. Using resolutions, the concept can be extended to coherent $\mathcal{D}$-modules. Almost elliptic complexes have a finite index. The computation of the index of D is reduced to that of D_ϵ and, thanks to [12], Louis obtains a formula which extends that of Atiyah–Singer and of Riemann–Roch.

6.2. THE RELATIVE INDEX FORMULA

Atiyah and Singer already gave a relative index formula, for elliptic systems depending on parameters, the "index" being a virtual fibre bundle on the space of parameters. The situation studied by Louis and Malgrange in [18] is much more general. What is given is a morphism $f: Y \to X$ of analytic varieties and a well-filtered coherent $\mathcal{D}_Y$-module P which satisfies an assumption of relative ellipticity, but which is not limited to derivatives along the fibres.

The direct image f_+P defined by Kashiwara is an element of the derived category of $\mathcal{D}_X$-modules. In this nonproper situation, a result of finiteness, due to Houzel and Schapira, asserts that f_+P is coherent. Moreover, if $Z \subset T^*Y$ contains the characteristic variety $\mathrm{char}(P)$, one can compute $Z' \subset T^*X$ which contains $\mathrm{char}(f_+P)$.

The equivalent of the index is constructed as follows. One can define the graduate $\mathrm{gr}(P)$ (the "symbol") associated with the good filtration. Then, one can associate with $\mathrm{gr}(P)$ an element of the Grothendieck group $[P]_Z^{\mathrm{an}} \in K_Z^{\mathrm{an}}$, whose image by the canonical homomorphism $K_Z^{\mathrm{an}} \to K_Z^{\mathrm{top}}$ is denoted by $[P]_Z^{\mathrm{top}}$.

The main result of the paper is a computation of $[f_+P]_{Z'}^{\mathrm{top}}$ starting from $[P]_Z^{\mathrm{top}}$. Actually, the former is the K-theoretic direct image of $[P]_Z^{\mathrm{top}}$ under (a compactification of) f.

There is also a statement in the real setting. In this case, Y is a real analytic manifold with boundary, and P should satisfy a relative almost ellipticity condition in Y and an extra condition on ∂Y, meaning that the boundary problem, without additional boundary condition, is elliptic. The proof is founded on the complex case, using small tubular neighbourhood of Y in its complexification.

7. Mathematical Physics

Mathematical physics had been a permanent concern for Louis. In 1979–1982, he organized, jointly with Adrien Douady and Jean-Louis Verdier, a seminar at the ENS [14] entitled “Mathematics and physics". He gave there talks on $\mathcal{D}$-modules and on the solution of the Riemann–Hilbert problem.

With Anne Boutet de Monvel and Gilles Lebeau [19], he studied small perturbations $H = H_0 + V$ of a harmonic oscillator $H_0 = -\Delta + \sum \lambda_j x_j^2$. Assuming $|D^\alpha V(x)| \lesssim |x|^{\gamma+\delta|\alpha|}$ with $\gamma + \delta < 1$, the authors prove that the Schwartz kernels of e^{itH} and e^{itH_0} have the same (modified) wave front set and thus that the singularities of $\mathrm{Tr}(e^{itH})$ are contained in the set of periods of the Hamiltonian flow of H_0. The proof uses nonstandard classes of operators which are microlocal and have good commutators, but are not pseudodifferential.

I still have to mention results with Iordan [20,25] on peak sets and articles with Chueshov, Khruslov and Rezounenko on several aspects of long-time evolution equations, existence of attractors and homogenization. Nonlinear parabolic (possibly retarded) equations [21,24], as well as models of diffusion of “coloured” (i.e. having different internal structure) particles which scatter and change “colour” on obstacles [22,23] are studied.

7.1. DEFORMATION QUANTIZATION

The notion of a star product is now a classical subject studied by many authors and naturally appear in various contexts. Two cornerstones of its history are first the papers of Bayen–Flato–Fronsdal–Lichnerowicz–Sternheimer in 1977 and that of Berezin in 1974 who define $\star$-products, and then the fundamental result of Kontsevich which, roughly speaking, asserts that any real Poisson manifold may be “quantized”, that is, endowed with a star algebra to which the Poisson structure is associated.

Louis was very much interested in this theory and made significant contributions. He was perhaps among the first ones to realize the importance of the theory of star products on *complex* Poisson manifolds. He enlarges a little the definition of star products to include the case of pseudodifferential operators as well as that of Toeplitz operators.

He gives in [26] see also [29] an almost complete classification of star algebras, as well as $\mathcal{E}$-algebras and $\mathcal{D}$-algebras (where $\mathcal{E}$ and $\mathcal{D}$ denote, respectively, the sheaf of rings of pseudodifferential and differential operators) on the cotangent space T^*X to a complex manifold, outside the zero section. He shows that the case where $\dim X \geq 3$ is "easy", contrarily to the cases where $\dim X$ is 1 or 2 where strange phenomena appear.

In [30], he discusses the case of star products of a given degree of homogeneity k. He shows that there are no nontrivial ones for $k \geq 3$, $k = 2$ produces just the Moyal star product and the case $k = 1$ is associated with a Lie algebra structure on the dual of the space.

In June 2014, in spite of his illness, Louis gave us a talk in the conference in honor of Gilles Lebeau. He studied the Wodzicki residual trace (noncommutative residue) of Toeplitz or pseudodifferential projectors and proved that this trace was always 0 for projectors of degree 0, but not if the degree was $\neq 0$. This is the latest in the long series of lectures by Louis that I have attended for about 50 years.

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